

# AN INTEGRAL REPRESENTATION FOR FOLLAND'S FUNDAMENTAL SOLUTION OF THE SUB-LAPLACIAN ON HEISENBERG GROUPS $\mathbb{H}^N$

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**ABSTRACT.** We prove that the Folland's fundamental solution for the sub-Laplacian on Heisenberg groups [2] can also be derived from the resolvent kernel of this sub-Laplacian [1]. This provides us with a new integral representation for this fundamental solution.

The Heisenberg group  $\mathbb{H}^n$  is the nilpotent Lie group whose underlying manifold is  $\mathbb{C}^n \times \mathbb{R}$  with coordinates  $(z, \tau) = (z_1, \dots, z_n; \tau)$  and whose group law is

$$(z, \tau) \cdot (w, s) = (z + w, \tau + s + 2\Im \langle z, w \rangle), \quad (1)$$

where  $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}$ . Letting  $z_j = x_j + iy_j \in \mathbb{C}$ , then,  $x_1, \dots, x_n; y_1, \dots, y_n$  and  $\tau$  are real coordinates on  $\mathbb{H}^n$ . We set

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right). \quad (2)$$

The operator

$$\Delta_{\mathbb{H}^n} := \sum_{j=1}^n \left[ -\frac{\partial^2}{\partial z_j \partial \bar{z}_j} - |z_j|^2 \frac{\partial^2}{\partial \tau^2} + i \frac{\partial}{\partial \tau} \left( z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) \right] \quad (3)$$

is left-invariant and is subelliptic of order  $\frac{1}{2}$  at each point  $(z, \tau)$  of  $\mathbb{H}^n$  (see Theorem 1 in [2]). Also, in [2], G.B. Folland has used the analogy with the fact that  $|x|^{2-m}$  is (a constant multiple of) the fundamental solution of Laplacian on  $\mathbb{R}^m$  with source at zero to prove that the sub-Laplacian in (3) admits a fundamental solution with source at zero of the form

$$G_0^F(z, \tau) = c_n \left( |z|^4 + \tau^2 \right)^{-\frac{n}{2}}; \quad (z, \tau) \in \mathbb{H}^n, \quad (4)$$

where

$$c_n = \frac{2^n \Gamma^2\left(\frac{n}{2}\right)}{\pi^{n+1}}. \quad (5)$$

In other words

$$\left( \Delta_{\mathbb{H}^n} [\phi], G_0^F \right) = \phi(0) \quad (6)$$

for any function  $\phi \in C_0^\infty(\mathbb{H}^n)$  (see [2, 3]).

Here, our aim is to derive the Folland's fundamental solution in (4) from the resolvent kernel of the sub-Laplacian in (3) on the Heisenberg group  $\mathbb{H}^n$ . This provides us with a new integral representation of  $G_0^F$ . Namely, we have the following

**Theorem.** *The Folland's fundamental solution in (4) can also be expressed as*

$$G_0^F(z, \tau) = \frac{2^{n+1} \Gamma\left(\frac{n}{2}\right)}{\pi^{n+1}} \int_0^{+\infty} x^{n-1} e^{-x|z|^2} \Psi\left(\frac{n}{2}, n; 2x|z|^2\right) \cos(\tau x) dx \quad (7)$$

in terms of the Tricomi  $\Psi$ -function.

To prove (7), we recall first that for  $\zeta \in \mathbb{C}$  with  $\Re(\zeta) < 0$ , the resolvent operator of  $\Delta_{\mathbb{H}^n}$  has the form

$$(\zeta - \Delta_{\mathbb{H}^n})^{-1}[\varphi](z, t) = \int_{\mathbb{H}^n} \mathcal{R}(\zeta; (z, \tau), (w, s)) \varphi(w, s) d\mu(w, s), \quad (8)$$

where the kernel function has been obtained in [1, p.4, Theorem 3.2] as

$$\begin{aligned} \mathcal{R}(\zeta; (z, \tau), (w, s)) &= \frac{-2^n}{\pi^{n+\frac{1}{2}}} \int_0^{+\infty} x^{n-1} \Gamma\left(\frac{n}{2} - \frac{\zeta}{2x}\right) \Psi\left(\frac{n}{2} - \frac{\zeta}{2x}, n; 2x|z-w|^2\right) \\ &\quad \times \exp\left(-x|z-w|^2\right) \cos(x(\tau-s) + 2x\Im\langle z, w \rangle) dx. \end{aligned} \quad (9)$$

The Tricomi  $\Psi$ -function can be defined as a linear combination of two  ${}_1F_1$ -sums ([5, p.56]):

$$\Psi(a, c; \zeta) := \left( \frac{\Gamma(1-c)}{\Gamma(a-c+1)} + \frac{\Gamma(c-1)}{\Gamma(a)} \zeta^{1-c} \right) {}_1F_1(a, c; \zeta), \quad (10)$$

where

$${}_1F_1(a, c; \zeta) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{j=0}^{+\infty} \frac{\Gamma(a+j)}{\Gamma(c+j)} \frac{\zeta^j}{j!}$$

is the confluent hypergeometric function defined with the usual condition  $c \neq 0, -1, -2, \dots$ . In the limit  $\zeta \rightarrow 0$  in (8), we obtain a right inverse of the operator in (3) as

$$\Delta_{\mathbb{H}^n}^{-1}[\varphi](z, \tau) = \int_{\mathbb{H}^n} \mathcal{R}(0; (z, \tau), (w, s)) \varphi(w, s) d\mu(w, s), \quad (11)$$

$d\mu$  being the Lebesgue measure on  $\mathbb{H}^n$ , or equivalently

$$\mathcal{R}_0 := -\mathcal{R}(0; (z, \tau), (w, s)) \quad (12)$$

is a Green kernel for  $\Delta_{\mathbb{H}^n}$  as pointed out in [1, p.6, Remark 3.3]. Now, to establish a connection between the integral kernel  $\mathcal{R}_0$  and the Folland's fundamental solution, we proceed by computing the integral

$$\begin{aligned} \mathcal{R}_0 &= \frac{2^n}{\pi^{n+\frac{1}{2}}} \int_0^{+\infty} x^{n-1} \Gamma\left(\frac{n}{2}\right) \Psi\left(\frac{n}{2}, n; 2x|z-w|^2\right) \\ &\quad \times \exp\left(-x|z-w|^2\right) \cos(x(\tau-s) + 2x\Im\langle z, w \rangle) dx. \end{aligned} \quad (13)$$

For this, let us rewrite (13) as

$$\mathcal{R}_0 = \frac{2^n}{\pi^{n+\frac{1}{2}}} \int_0^{+\infty} x^{n-1} \left[ \Gamma\left(\frac{n}{2}\right) \Psi\left(\frac{n}{2}, n; \mu x\right) \right] e^{-\frac{\mu}{2}x} \cos(x\theta) dx, \quad (14)$$

where we have set

$$\mu := 2|z-w|^2 \quad \text{and} \quad \theta = (\tau-s) + 2\Im\langle z, w \rangle. \quad (15)$$

Following Tricomi [5, p.90], there is an integral representation for the  $\Psi$ -function in (10) of the form:

$$\Gamma(a) \Psi(a, c; u) = \int_0^{+\infty} e^{-at} \exp\left(-\frac{u}{e^t-1}\right) (1-e^{-t})^{-c} dt. \quad (16)$$

For the parameters  $a = \frac{n}{2}$ ,  $c = n$  and  $u = \mu x$ , the integral in (14) takes the form

$$\mathcal{R}_0 = \frac{2^n}{\pi^{n+\frac{1}{2}}} \int_0^{+\infty} x^{n-1} \left[ \int_0^{+\infty} e^{-\frac{n}{2}t} \exp\left(-\frac{\mu x}{e^t - 1}\right) (1 - e^{-t})^{-n} dt \right] e^{-\frac{\mu}{2}x} \cos(x\theta) dx. \quad (17)$$

Intertwining the integrals, we rewrite (17) as

$$\mathcal{R}_0 = \frac{2^n}{\pi^{n+\frac{1}{2}}} \int_0^{+\infty} \frac{e^{-\frac{n}{2}t}}{(1 - e^{-t})^n} \left[ \int_0^{+\infty} x^{n-1} \exp\left(-\frac{\mu x}{2} \coth\left(\frac{t}{2}\right)\right) \cos(x\theta) dx \right] dt. \quad (18)$$

To calculate the integral

$$\int_0^{+\infty} x^{n-1} \exp\left(-\frac{\mu x}{2} \coth\left(\frac{t}{2}\right)\right) \cos(x\theta) dx, \quad (19)$$

we make appeal to the identity ([4, p.498])

$$\int_0^{+\infty} x^{\nu-1} e^{-\alpha x} \cos(\theta x) dx = \Gamma(\nu) (\alpha^2 + \theta^2)^{-\frac{\nu}{2}} \cos\left(\nu \arctan\left(\frac{\theta}{\alpha}\right)\right) \quad (20)$$

when  $\Re(\nu) > 0$ ,  $\Re(\alpha) > |\Im(\theta)|$  are fulfilled. In our case  $\nu = n$  and  $\alpha = \frac{\mu}{2} \coth\left(\frac{t}{2}\right)$ , and therefore (18) reads

$$\begin{aligned} \mathcal{R}_0 = \frac{2^n \Gamma(n)}{\pi^{n+\frac{1}{2}}} \int_0^{+\infty} \frac{e^{-\frac{n}{2}t}}{(1 - e^{-t})^n} & \left( \left( \frac{\mu}{2} \coth\left(\frac{t}{2}\right) \right)^2 + \theta^2 \right)^{-\frac{n}{2}} \\ & \times \cos\left(n \arctan\left(\frac{\theta}{\frac{\mu}{2} \coth\left(\frac{t}{2}\right)}\right)\right) dt. \end{aligned} \quad (21)$$

After some calculations, we see that equation (21) takes the form

$$\begin{aligned} \mathcal{R}_0 = \frac{\Gamma(n)}{\pi^{n+\frac{1}{2}}} \int_0^{+\infty} & \left( \sinh\left(\frac{t}{2}\right) \right)^{-n} \left( \frac{\mu}{2} \coth\left(\frac{t}{2}\right) \right)^{-n} \\ & \times \left( 1 + \left( \frac{\theta}{\frac{\mu}{2} \coth\left(\frac{t}{2}\right)} \right)^2 \right)^{-\frac{n}{2}} \cos\left(n \arctan\left(\frac{\theta}{\frac{\mu}{2} \coth\left(\frac{t}{2}\right)}\right)\right) dt. \end{aligned} \quad (22)$$

Making the change variable  $\rho = \frac{\theta}{\frac{\mu}{2} \coth\left(\frac{t}{2}\right)}$ , then (22) becomes

$$\mathcal{R}_0 = \frac{2^n \Gamma(n)}{\mu^{n-1} \theta \pi^{n+\frac{1}{2}}} \int_0^{\frac{2\theta}{\mu}} \left( 1 - \left( \frac{\mu}{2\theta} \right)^2 \rho^2 \right)^{\frac{n}{2}-1} (1 + \rho^2)^{-\frac{n}{2}} \cos(n \arctan(\rho)) d\rho. \quad (23)$$

A second change of variable,  $\arctan(\rho) = u$ , shows that (23) can be reduced to

$$\mathcal{R}_0 = \frac{2^n \Gamma(n)}{\mu^{n-1} \theta \pi^{n+\frac{1}{2}}} \int_0^{\arctan\left(\frac{2\theta}{\mu}\right)} \left( \cos^2(u) - \left( \frac{\mu}{2\theta} \right)^2 \sin^2 u \right)^{\frac{n}{2}-1} \cos(nu) du. \quad (24)$$

Next, by a trigonometrical linearization, we can rewrite (24) as

$$\begin{aligned} \mathcal{R}_0 &= \frac{2^{\frac{n}{2}+1}\Gamma(n)}{\mu^{n-1}\theta\pi^{n+\frac{1}{2}}} \left(1 + \left(\frac{\mu}{2\theta}\right)^2\right)^{\frac{n}{2}-1} \\ &\quad \times \int_0^{\arctan(\frac{2\theta}{\mu})} \left(\cos 2u - \frac{1 - \left(\frac{2\theta}{\mu}\right)^2}{1 + \left(\frac{2\theta}{\mu}\right)^2}\right)^{\frac{n}{2}-1} \cos(nu) du. \end{aligned} \quad (25)$$

Setting  $\beta = \left(\frac{2\theta}{\mu}\right)^2$  and making the change of variable  $2u = \kappa$  in (25) give that

$$\mathcal{R}_0 = \frac{2^{\frac{n}{2}}\Gamma(n)}{\mu^{n-1}\theta\pi^{n+\frac{1}{2}}} \left(\frac{\beta+1}{\beta}\right)^{\frac{n}{2}-1} \int_0^{2\arctan(\frac{2\theta}{\mu})} \left(\cos(\kappa) - \frac{1-\beta}{1+\beta}\right)^{\frac{n}{2}-1} \cos\left(\frac{n}{2}\kappa\right) d\kappa. \quad (26)$$

Now, by putting

$$\cos(\varepsilon) = \frac{1-\beta}{1+\beta}, \quad (27)$$

we can check that

$$\varepsilon = \arccos\left(\frac{1-\beta}{1+\beta}\right) = \arccos\left(\frac{1 - \left(\frac{\mu}{2\theta}\right)^2}{1 + \left(\frac{\mu}{2\theta}\right)^2}\right) = 2\arctan\left(\frac{2\theta}{\mu}\right). \quad (28)$$

We are now in position to apply the identity ([4, p.406]):

$$\int_0^\varepsilon (\cos(x) - \cos(\varepsilon))^{\nu-\frac{1}{2}} \cos(ax) dx = \sqrt{\frac{\pi}{2}} (\sin(\varepsilon))^\nu \Gamma\left(\nu + \frac{1}{2}\right) P_{a-\frac{1}{2}}^{-\nu}(\cos(\varepsilon)) \quad (29)$$

with the conditions  $\Re(\nu) > -\frac{1}{2}$ ,  $a > 0$  and  $0 < \varepsilon < \pi$  and where

$$P_\lambda^\nu(x) := \frac{1}{\Gamma(1-\nu)} \left(\frac{x+1}{x-1}\right)^{\nu/2} {}_2F_1\left(-\lambda, \lambda+1; 1-\nu; \frac{1-x}{2}\right) \quad (30)$$

denotes the Legendre function of the first kind [4, p.959]. In our setting, the integral in (26) reads

$$\begin{aligned} &\int_0^{2\arctan(\frac{2\theta}{\mu})} \left(\cos(\kappa) - \frac{1-\beta}{1+\beta}\right)^{\frac{n-1}{2}-\frac{1}{2}} \cos\left(\frac{n}{2}\kappa\right) d\kappa \\ &= \sqrt{\frac{\pi}{2}} (\sin(\varepsilon))^{\frac{n-1}{2}} \Gamma\left(\frac{n}{2}\right) P_{\frac{n-1}{2}}^{-\left(\frac{n-1}{2}\right)}(\cos(\varepsilon)). \end{aligned} \quad (31)$$

Hence, using the Gegenbauer representation ([4, p.969]):

$$P_\sigma^{-\sigma}(\cos(\varepsilon)) = \frac{1}{\Gamma(1+\sigma)} \left(\frac{1}{2} \sin(\varepsilon)\right)^\sigma \quad (32)$$

with  $\sigma = \frac{n-1}{2}$ , we can write the right hand side in (31) as

$$\begin{aligned} \sqrt{\frac{\pi}{2}} (\sin(\varepsilon))^{\frac{n-1}{2}} \Gamma\left(\frac{n}{2}\right) P_{\frac{n-1}{2}}^{-\left(\frac{n-1}{2}\right)}(\cos(\varepsilon)) &= \left(\frac{1}{2}\right)^{\frac{n-1}{2}} \sqrt{\frac{\pi}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} (\sin(\varepsilon))^{n-1} \\ &= \left(\frac{1}{2}\right)^{\frac{n}{2}} \sqrt{\pi} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} (1 - \cos^2(\varepsilon))^{\frac{n-1}{2}}. \end{aligned} \quad (33)$$

Returning back to (26), keeping in mind the expression of  $\varepsilon$  given through (28), we obtain that

$$\mathcal{R}_0 = \frac{2^{\frac{n}{2}} \Gamma(n)}{\mu^{n-1} \theta \pi^{n+\frac{1}{2}}} \left( \frac{\beta+1}{\beta} \right)^{\frac{n}{2}-1} \left( \left( \frac{1}{2} \right)^{\frac{n}{2}} \sqrt{\pi} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \left( \frac{4\beta}{(1+\beta)^2} \right)^{\frac{n-1}{2}} \right) \quad (34)$$

$$= \frac{2^{n-1}}{\mu^{n-1} \theta \pi^n} \frac{\Gamma(n) \Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \left( \frac{\beta+1}{\beta} \right)^{-\frac{n}{2}}. \quad (35)$$

Now, replacing  $\beta$  by  $\left( \frac{2\theta}{\mu} \right)^2$  and using the expressions of  $\mu$  and  $\theta$  as in (15), we arrive at

$$\mathcal{R}_0 = \frac{\Gamma(n) \Gamma(\frac{n}{2})}{\pi^n \Gamma(\frac{n+1}{2})} \left( (|z-w|^2)^2 + ((\tau-s) + 2\Im \langle z, w \rangle)^2 \right)^{-\frac{n}{2}}. \quad (36)$$

$$= \frac{2^{n-1} \Gamma^2(\frac{n}{2})}{\pi^{n+\frac{1}{2}}} \left( (|z-w|^2)^2 + ((\tau-s) + 2\Im \langle z, w \rangle)^2 \right)^{-\frac{n}{2}}. \quad (37)$$

The last equality follows using Legendre's duplication formula ([4, p.896]):

$$\Gamma(\xi) \Gamma\left(\xi + \frac{1}{2}\right) = 2^{1-2\xi} \sqrt{\pi} \Gamma(2\xi)$$

for  $\xi = \frac{n}{2}$ . Therefore, we assert that

$$\mathcal{R}_0 = \frac{2^{n-1} \Gamma^2(\frac{n}{2})}{\pi^{n+\frac{1}{2}} c_n} G_0^F \left( (z, \tau) \circ (w, s)^{-1} \right), \quad (38)$$

where the constant  $c_n$  is as in (5). In particular, for  $(w, s) = (0, 0)$ , Equation (38) reduces further to

$$\mathcal{R}_0 = \frac{2^{n-1} \Gamma^2(\frac{n}{2})}{\pi^{n+\frac{1}{2}} c_n} G_0^F(z, \tau) = \frac{\sqrt{\pi}}{2} G_0^F(z, \tau). \quad (39)$$

Finally, by combining (14) and (39), we get the announced result of the theorem.

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